Objects of the experiment

- Determining the moment of inertia of a circular disk for various distances between the axis of rotation and the axis of symmetry.
- Confirming Steiner's theorem (parallel axis theorem).

Principles

The moment of inertia of an arbitrary rigid body whose mass elements \( \Delta m \) have the distances \( r_i \) from the axis of rotation \( A \) is

\[
J_A = \sum_i \Delta m_i \cdot r_i^2 \quad (I).
\]

If the axis of rotation does not pass through the centre of mass of the body, application of Eq. (I) leads to an involved calculation. Often it is easier to calculate the moment of inertia \( J_S \) with respect to the axis \( S \), which is parallel to the axis of rotation and passes through the centre of mass of the body.

For deriving the relation between \( J_A \) and \( J_S \), the plane perpendicular to the axis of rotation where the respective mass element \( \Delta m \) is located is considered (see Fig. 1). In this plane, the vector \( a \) points from the axis of rotation to the centre-of-mass axis, the vector \( r_i \) points from the axis of rotation to the mass element \( \Delta m_i \), and the vector \( s_i \) points from the centre-of-mass axis to the mass element. Thus

\[
r_i = a + s_i \quad (II),
\]

and the squares of the distances in Eq. (I) are

\[
r_i^2 = (a + s_i)^2 = a^2 + 2 \cdot a \cdot s_i + s_i^2 \quad (III).
\]

Therefore the sum in Eq. (I) can be split into three terms:

\[
J = \sum_i \Delta m_i \cdot a^2 + 2 \sum_i \Delta m_i \cdot s_i \cdot a + \sum_i \Delta m_i \cdot s_i^2 \quad (IV).
\]

In the first summand,

\[
\sum_i \Delta m_i = M
\]

is the total mass of the body. In the last summand,

\[
\sum_i \Delta m_i \cdot s_i^2 = J_S
\]

is the moment of inertia of the body with respect to the centre-of-mass axis. In the middle summand,

\[
\sum_i \Delta m_i \cdot s_i = 0
\]

because the vectors \( s_i \) start from the axis through the centre of mass.

Thus Steiner's theorem follows from Eq. (IV):

\[
J_A = M \cdot a^2 + J_S \quad (V)
\]

This theorem will be verified in the experiment with a flat circular disk as an example. Its moment of inertia \( J_A \) with respect to an axis of rotation at a distance \( a \) from the axis of symmetry is obtained from the period of oscillation \( T \) of a torsion axle to which the circular disk is attached. We have

\[
J_A = D \left( \frac{T}{2\pi} \right)^2 \quad (VI)
\]

\( D \): restoring torque of the torsion axle

Fig. 1  Schematic illustration referring to the derivation of Steiner's theorem (parallel axis theorem)
**Apparatus**

1. torsion axle 347 80
2. circular disk for the torsion axle 347 83
3. stand base, V-shape, 20 cm 300 02
4. stopclock I, 30 s / 15 min 313 07

**Setup and carrying out the experiment**

The experimental setup is illustrated in Fig 2.

- Fix the centre of the circular disk to the torsion axle and mark the equilibrium position on the table.
- Rotate the circular disk by 180° from the equilibrium position and release it.
- Start the time measurement as soon as the circular disk passes through the equilibrium position and stop the measurement after five oscillations.
- Repeat the measurement four times alternately deflecting the disk to the left and to the right.
- Calculate the period of oscillation $T$ from the mean value of the five measurements.
- Mount the circular disk on the torsion axle so that its centre is at a distance of 2 cm from the axle, and, if necessary, mark the equilibrium position anew.
- Measure the time of five oscillations five times alternately deflecting the disk to the right and to the left.
- Calculate the period of oscillation $T$.
- Repeat the measurement for other distances $a$ from the axis of symmetry.

**Measuring example**

Mass of the disk: $M = 704$ g
Radius of the disk: $R = 20$ cm

Table 1: measured time of five oscillations for various distances $a$ between the axis of rotation and the axis of symmetry and oscillation periods $T$ calculated from the mean value of the measured values

<table>
<thead>
<tr>
<th>$a$ (cm)</th>
<th>$5 \cdot T$ (s)</th>
<th>$T$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24.7</td>
<td>24.7</td>
</tr>
<tr>
<td>2</td>
<td>25.0</td>
<td>24.7</td>
</tr>
<tr>
<td>4</td>
<td>25.5</td>
<td>26.7</td>
</tr>
<tr>
<td>6</td>
<td>26.7</td>
<td>26.7</td>
</tr>
<tr>
<td>8</td>
<td>28.1</td>
<td>28.1</td>
</tr>
<tr>
<td>10</td>
<td>29.9</td>
<td>30.0</td>
</tr>
<tr>
<td>12</td>
<td>32.6</td>
<td>32.6</td>
</tr>
<tr>
<td>14</td>
<td>35.0</td>
<td>35.2</td>
</tr>
<tr>
<td>16</td>
<td>37.3</td>
<td>37.3</td>
</tr>
</tbody>
</table>

**Evaluation**

With the aid of Eq. (VI), the moment of inertia $J_A$ can be calculated from the values of the period of oscillation listed in Table 1. The restoring torque $D$ required for the calculation was determined in the experiment P1.4.5.1:

$$D = 0.023 \text{Nm rad}^{-1}$$

The results are listed in Table 2.

Table 2: list of the squares $a^2$ of the distance $a$ and the moments of inertia $J_A$ calculated from the period of oscillation $T$

<table>
<thead>
<tr>
<th>$a$ (cm)</th>
<th>$a^2$ (cm²)</th>
<th>$(\frac{T}{2\pi})^2$ (s²)</th>
<th>$\frac{J_A}{g \cdot m^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.610</td>
<td>14.0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.618</td>
<td>14.2</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>0.667</td>
<td>15.3</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td>0.723</td>
<td>16.6</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>0.809</td>
<td>18.6</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>0.916</td>
<td>21.1</td>
</tr>
<tr>
<td>12</td>
<td>144</td>
<td>1.074</td>
<td>24.7</td>
</tr>
<tr>
<td>14</td>
<td>196</td>
<td>1.253</td>
<td>28.8</td>
</tr>
<tr>
<td>16</td>
<td>256</td>
<td>1.439</td>
<td>33.1</td>
</tr>
</tbody>
</table>
Fig. 3 Moment of inertia $J_A$ as a function of the square of the distance $a$ between the axis of rotation and the axis of symmetry

Eq. (V) describes a linear relation between $J_A$ and $a^2$ with the slope $M$ and the intercept of the ordinate $J_S$. This relation is confirmed by Fig. 3. The straight line drawn in the figure has the slope $M = 740 \text{ g}$ and the intercept of the ordinate $J_s = 14 \text{ g m}^2$.

Result

The moment of inertia of a body with respect to an arbitrary axis can be calculated from the moment of inertia with respect to the centre-of-mass axis, the total mass and the distance between the two axes.